A Brief Introduction to the Theory of Wavelets

by

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Much of today's applications of mathematics to engineering problems originate in mathematical theories developed in the last century or before. Mathematics has traditionally been well ahead of its applications. This is because conceptual ideas - such as those needed to compute and categorize various quantities (signals, data, etc.) - can leap far ahead of the technology available to implement them. Computers have now changed that. Computational technology has caught up with the ideas and, though there are several examples of this, wavelets are by far the most striking.

Wavelets are a mathematical tool that are the outcome of over fifty years effort to deal with one of the most intractable obstacles to mathematical, physical and engineering analysis. This obstacle is known as the (Heisenberg) Uncertainty Principle. It refers to the inability - dictated by the laws of nature - to know exactly both the location and degree of change in a quantity which is varying with time. The following simple example illustrates it.

Suppose a trumpet player plays a low note for a while and then suddenly switches to a higher one, which he plays for some time. We want to analyze the signal. By sampling the signal in time we can determine fairly accurately when the change occurred. But the abruptness of the change - especially if it is sharp - will be more difficult to determine. (Was it an instantaneous jump? Did it occur slowly over one time interval? etc.) If, instead of sampling the signal directly, we sample the amplitudes of various frequency components of the signal - that is, we consider the Fourier Series or Fourier Transform of the signal - the abruptness of the change will be evident from the amplitudes of the high frequency components of the signal. But now we will have less accurate knowledge of the time of the change. That is because each frequency component of a signal is a single frequency wave that is infinite in duration. Waves of very high frequency will tell about the abruptness of the change, but, by themselves, will give us no information about the location in time of the change. For that, we will have to add all the waves together to reconstruct the signal as it occurred in time. (That these infinite-duration, single-frequency waves can theoretically be added together to reconstruct the original signal is the grand discovery of the French mathematician Joseph Fourier in 1803 and is at the heart of the most powerful tool of modern engineering, Fourier Analysis.) Since our sampling focused entirely on frequencies, the reconstruction of our original signal in time will not be as accurate as the time sampled version. Thus, we will lose accuracy about the exact location of the change. In conclusion, if we sample our wave directly in time then we lose information about the frequency behavior of our signal; if we sample the frequencies then we may lose information about the time behavior of the signal. This is the uncertainty principle in action.

Wavelet analysis addresses this problem directly. Fourier showed that all signals (of interest) can be reconstructed from single-frequency, infinite-duration waves and that these waves form a minimal set for the reconstruction - that is, none of the waves are redundant. (A collection of functions (or signals) with that can be used to represent other signals in a non-redundant way is called a basis.) In Fourier Analysis, the basis functions are generally sines and cosines of varying frequencies.

In 1985 another French mathematician, Yves Myer, showed that all signals (of interest) can be reconstructed from wavelets, and that these wavelets form a minimal set for the reconstruction. In other words, he showed that wavelets form a basis. Also, he showed that wavelet analysis has many useful properties that Fourier Analysis does not have.

In Fourier analysis the signal is thought of as consisting of a weighted sum of single-frequency, infinite-duration waves. The weights are called coefficients. The analysis consists of determining the coefficients. In wavelet analysis, the signal is represented as a weighted sum of wavelets. Each wavelet is a packet of oscillations that occur within a fixed interval of time and have their frequencies concentrated within a fixed bandwidth. (See the attached figures.) Thus, the amount of a wavelet present in a signal - given by its coefficient - represents a specific highly concentrated range of frequencies of the signal contained within a fixed time duration. This is the critical property of wavelets. Thus, wavelets deal directly with the problems of the uncertainty principle. In the trumpet example above, a high frequency wavelet centered in a specific interval of time would have a large coefficient if the trumpeter changed notes abruptly in that interval.

Wavelet analysis has other advantages over Fourier Analysis. One is that wavelets can be made to have properties that better suit the analysis to be undertaken. That is because there are many different wavelet bases. (In fact, the number of bases is unlimited.)

Wavelet bases are constructed from a fixed function and it is informative to see how this is done. One starts with a single wavelet. New wavelets are produced by copying this wavelet onto regularly spaced intervals on the time axis. It is then "squeezed" which increases its oscillations while making its duration smaller. (See the attached figures.) Copies of this "squeezed" wavelet are centered on smaller regularly spaced intervals. The process is continued. Finally, by starting with the original wavelet and "stretching" it, wavelets with lower frequencies are copied to larger and larger intervals. It is easy to see that the choice of the original wavelet determines the qualities of the particular collection.

(Algorithms have been developed that both achieve the process just described and take the wavelet transform of a signal, but in a much more efficient way.)

The idea of an adaptable collection of functions (or wavelets) for the analysis of signals has recently led researchers to develop a method for even mixing various collections in order to arrive at the collection best suited to an application. These ideas have been used to study voice analyses and compression.

In addition to adaptability, there is another very important feature of wavelet analysis that is in stark contrast to Fourier Analysis. The Fourier coefficients of a signal do not contain very good qualitative information about many aspects of the signal, such as its smoothness. Wavelet coefficients do contain this information. For example, the coefficients decrease rapidly if the signal is smooth. This is very important, because this means that the smoothness of the signal helps determine the rate at which it can be reconstructed by wavelets.

An additional aspect of wavelets should be noted. Fourier Analysis not only simplifies the processing of signals, but it also simplifies the analysis of various systems - including systems that act on these signals. (This is another major reason for its widespread use in engineering.) Wavelets also allow this analysis.

In the beginning, I said that wavelets are the most striking example of the value to engineering of the tools of modern mathematical research. Wavelets themselves are the most recent development in two mathematical theories, Littlewood-Paley Theory - about 65 years old - and Calderon-Zygmund Theory - about 30 years old. Within the past twenty years, other tools, similar to wavelets, have appeared in the work of mathematicians specializing in these theories. These tools have also been used to analyze signals and systems - though not in actual practice. I believe that these tools can also be fruitfully utilized in practical signal and system analysis - again because modern technology can now be used to implement such a detailed analyses.