High School Physics, Radians, and a Proof that $(\sin(ax))' = a\cos(ax)$.

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The proof of the identity in the title usually presented to an elementary calculus class involves four steps: i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$, ii) $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$, iii) the addition formula for sin and cos and iv) the chain rule. This has pedagogical limitations. It makes the derivation of the equation seem like mathematical magic involving manipulation of trigonometric identities and theorems; it tends to obscure the natural relationship between the sin and cos functions; and, finally, by the time the complete formula is developed, the student may well have forgotten **why** radian measure is important. In this note we present a method that (in our experience) overcomes these difficulties by appealing to the students' physical intuition.

We begin by reminding the student that a function can be thought of as the position of a particle and the derivative as the velocity. We then consider a particle moving in a counterclockwise direction around the circumference of the unit circle with constant speed one. We let $f(t)$ be the y component of the position vector. (We think of shining lights from right to left and observing the motion of the shadow.) Since the particle is revolving around a unit circle with unit speed, the time is the same as the angle that the position vector makes with the horizontal **if** we use radians. Therefore, $f(t) = \sin(t)$ and $f'(t)$ is the y-component of the velocity vector. From elementary geometry we get $f'(t) = \cos t$ (see figure 1). Finally, to see that $(\sin(at))' = a\cos(at)$, one need only change the radius of the circle or the magnitude of the velocity vector. The derivative of the cos can be similarly determined by letting f(t) be the x-component of the position vector.

Remark: Since $\lim_{x \to 0} \frac{\sin x}{x} = (\sin x)'(0)$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = (\cos x)'(0)$ it is a simple corollary that these limits are one and zero respectively. It would be interesting to see if the chain rule or the addition formulas for sin and cos could be derived from the geometric arguments in this note.