A NOTE ON BERGMAN SPACES

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Abstract. Let $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y>0\}$ and let $R_1, \dots, R_n$ be the Riesz transforms. We give an elementary proof of a theorem of T.M. Flett: Let $0 < p \le 1$ and $\beta > -1$. There is a $C_{p,n,\beta}$ such that for u harmonic on $\mathbb{R}^{n+1}_+$

$$\int_{\mathbb{R}^{n+1}_+} \sum_1^n |R_i u|^p y^\beta \, dx\, dy \le C_{p,n,\beta} \int_{\mathbb{R}^{n+1}_+} |u|^p y^\beta \, dx\, dy$$

We show that $\beta > -1$ is sharp.

We obtain as a corollary (of our proof) that certain distributions which have "atomic decompositions" can always in terms of smooth (C^1) atoms.

... $R_1, \ldots, R_n$
be the Riesz transforms. We prove the following:

<u>Theorem 1</u> Let $0 < p \le 1$ and $\beta > -1$. There is a constant $C_{pn\beta}$ such that if u is harmonic on $\mathbb{R}^{n+1}_+$ then:

$$1) \qquad \int_{\mathbb{R}^{n+1}_+} \sum_{i=1}^{n} |R_i u|^p y^\beta \, dx\, dy \le C_{pn\beta} \int_{\mathbb{R}^{n+1}_+} |u|^p y^\beta \, dx\, dy$$

This theorem seems to have been first proved by Flett [ ]. We give an elementary proof.

After announcing our result, we learned that it in some measure parallels recent work of Ricci and Taibleson [ ] [ ].

We are grateful to Joel Shapiro and David Ullrich for helpful communications.

Theorem 1 is clearly equivalent to:

<u>Theorem 2</u> Let $0 < p \le 1$, $\beta > -1$. If U is harmonic on $\mathbb{R}^{n+1}_+$ and $|\nabla U| \to 0$ as $y \to \infty$ then:

$$\int |\nabla U|^p y^\beta \, dx\, dy \le C_{pn\beta} \int \left|\frac{\partial U}{\partial y}\right|^p y^\beta \, dx\, dy$$

We shall prove Theorem 2 by means of an "atomic decomposition".

Definition. For $0<p\le 1$, $\beta>-1$, we say that $b(x)\in L^1(\mathbb{R}^n)$ is an (n,p,β)-atom if:

i) $\operatorname{supp} b\subset Q$, Q a cube,

ii) $\|b\|_\infty \le \ell(Q)^{1-\frac{n+1+\beta}{p}}$ ($\ell(Q)$ is the sidelength of Q).

iii) $\int b(x)\,x^\alpha\,dx=0$ for all monomials x^α with $|\alpha|\le\left[\frac{n+1+\beta}{p}-(n+1)\right]$.

If $\beta<0$, or $p=1$ and $\beta=0$, then $b(x)$ also satisfies:

iv) $\|\nabla b\|_\infty \le \ell(Q)^{-\frac{n+1+\beta}{p}}$.

Let P_y be the Poisson kernel. A computation shows that if $u=P_y * b$, b an (n,p,β)-atom, then:

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u|^p\, y^\beta\, dx\,dy \le C$$

thus, modulo a limiting argument (which we give at the end), Theorem 2 will follow from:

Lemma 1 Let $0<p\le 1$, $\beta>-1$. Let $f\in L^\infty(\mathbb{R}^n)$ and $|\nabla f|\in L^\infty(\mathbb{R}^n)$, and set $u=P_y * f$. If:

$$\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial u}{\partial y}\right|^p y^\beta\,dx\,dy<\infty$$

then there exist numbers λ_k and (n,p,β)-atoms a_k so that:

$$f=\sum \lambda_k a_k + (\text{const}) \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

with:

$$\sum|\lambda_k|^p \le C\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial u}{\partial y}\right|^p y^\beta\,dx\,dy$$

... is Hardy and Littlewood: Let $0 < p \le 1$. Then is a C_{pn} so that if u is harmonic on $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ then:

$$2) \qquad |u(0)| \le \left(\frac{C_{pn}}{R^n} \int_{B_R} |u(x)|^p \, dx \right)^{1/p}$$

For the beautiful proof, see [1].

<u>Proof of Lemma 1.</u> Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be real, radial, supp $\varphi \subset \{|x| < 1\}$, and let φ have the cancellation property iii). Normalize φ so that:

$$\int_0^\infty e^{-\theta} \hat{\varphi}(\theta, 0, 0, \ldots, 0) \, d\theta = -1$$

Let $\varphi_y(x) = y^{-n} \varphi\left(\frac{x}{y}\right)$. By Fourier transforms:

$$f = \int_{\mathbb{R}^{n+1}_+} \frac{\partial u}{\partial y}(t, y) \varphi_y(x - t) \, dt \, dy + (\text{const})$$

We [illegible] henceforth) assume that

(const) $= 0$.

For each dyadic cube $Q \subset \mathbb{R}^n$ define:

$$Q^+ = \left\{ (x,y) : \ x \in Q, \ \frac{l(Q)}{2} < y \leq l(Q) \right\}$$

Set:

$$g_Q = \int_{Q^+} \frac{\partial u}{\partial y}(t,y)\, \varphi_y(x-t)\, dt\, dy$$

Then:

$$f = \sum g_Q \quad \text{in } \mathcal{S}'$$

Let $L(Q^+)$ be a $(1+\varepsilon)$-expansion of Q^+ ($\varepsilon > 0$ small and fixed). Let C_1 be large and fixed and define numbers:

$$\lambda_Q = \left(\int_{L(Q^+)} \left| \frac{\partial u}{\partial y} \right|^p y^\beta \, dx\, dy \right)^{\frac{1}{p}}$$

Clearly:

$$\sum \lambda_Q^p \leq C \iint_{\mathbb{R}^{n+1}_+} \left|\frac{\partial u}{\partial y}\right|^p y^\beta \, dx\, dy$$

We claim that the $a_Q = g_Q/\lambda_Q$ are (n,p,β)-atoms. Properties i) and (iii) are obvious, and iv) can be proved in the same manner as ii). Thus, it is enough to estimate $\|g_Q\|_\infty$.

For each $(t,y) \in Q^+$, inequality 2) gives:

$$\left|\frac{\partial u}{\partial y}(t,y)\right| \leq \left(\frac{C_{p\varepsilon}}{\ell(Q)^{n+1}} \iint_{L(Q^+)} \left|\frac{\partial u}{\partial \eta}\right|^p d\xi\, d\eta\right)^{1/p}$$

$$\leq \frac{c\,\lambda_Q}{\ell(Q)^{\frac{n+1+\beta}{p}}}$$

Therefore:

$$|g_Q| \leq \frac{c\,\lambda_Q}{\ell(Q)^{\frac{n+1+\beta}{p}}} \cdot \frac{\ell(Q)^{n+1}}{\ell(Q)^{n}}$$

$$\leq c\,\lambda_Q \cdot \ell($$

Lemma 1 is proved. ///

The limiting argument consists of the following two lemmas.

Lemma 2. Let $0<p\leq 1$, $\beta>-1$. If u is harmonic on $\mathbb{R}^{n+1}_+$ and:

$$3)\qquad \int_{\mathbb{R}^{n+1}_+} |u|^p y^\beta \, dx\, dy < \infty$$

then for each $y_0>0$ the function:

$$f^{y_0}(x) = \int_{y_0}^{\infty} u(x,y)\, dy$$

is bounded and has bounded gradient.

Lemma 3. With u, p, β as above:

$$\lim_{y_0\to 0} \int_{y>y_0} |u|^p (y-y_0)^\beta \, dx\, dy = \int_{\mathbb{R}^{n+1}_+} |u|^p y^\beta \, dx\, dy$$

Proof of Lemma 2. By considering the

in $\mathbb{R}_+$, centered at (x,y), of radius $y/2$, inequality 2)

yields:

$$|u(x,y)| \leq \frac{C}{y^{\frac{n+1+\beta}{p}}} \tag{4}$$

But $\frac{n+1+\beta}{p} > 1$. Thus $f^{y_0} \in L^\infty(\mathbb{R}^n)$. It has bounded gradient since $f^{y_0} = P_{y_0/2} * f^{y_0/2}$. ///

Proof of Lemma 3. It is enough to show:

$$\lim_{y_0 \to 0} \int_{y_0 < y < 2y_0} |u|^p (y-y_0)^\beta \, dx\,dy = 0$$

Let $\varepsilon > 0$ be small and define:

$$\Sigma'(y_0) = \left\{ (x,y) : (1-\varepsilon)y_0 < y < (2+\varepsilon)y_0 \right\}$$

$$\Delta(y_0) = \int_{\Sigma'(y_0)} |u|^p y_0^\beta \, dx\,dy$$

Clearly $\Delta(y_0) \to 0$ as $y_0 \to 0$.

For $(x,y) \in \mathbb{R}^{n+1}_+$, $y_0 < y < ?$

be the ball centered at (x,y) of radius εy_0. Inequality 2) yields, for such (x,y):

$$|u(x,y)|^p \le \frac{C}{y_0^{n+1}} \int_{D(x,y)} |u(\xi,\eta)|^p d\xi\, d\eta$$

Now, fixing y, Fubini's theorem implies:

$$\int |u(x,y)|^p dx \le \frac{C}{y_0^{1+\beta}} \Delta(y_0)$$

Therefore:

$$\int_{y_0<y<2y_0} |u|^p (y-y_0)^\beta dx\, dy \le \frac{C\Delta(y_0)}{y_0^{1+\beta}} \int_0^{y_0} y^\beta dy$$

$$\le C\Delta(y_0) \longrightarrow 0$$

since $\beta>-1$. QED ///

The condition $\beta>-1$ is best

serve that $\beta \le -1$ makes the

or trivial. To see this, assume 1) holds ~~…~~

~~Consider two cases~~ (with $u \not\equiv 0$) and consider two cases:

Case I. $-n-1 \leq \beta \leq -1$. Let R_0 denote the identity operator. For each $(k) = (k_1, \ldots k_L)$ $(0 \leq k_i \leq n)$ define:

$$u_{(k)} = R_{k_1} \cdots R_{k_L} u$$

For each L define:

$$F_L(x,y) = \Big(\sum_{\substack{\text{all such}\\(k)}} |u_{(k)}|^2 \Big)^{p/2}$$

It is well known that if $L > N(n,p)$ then F_L is subharmonic [1]. If 1) were true, we would have:

$$5) \qquad \int_{\mathbb{R}^{n+1}_+} F_L \, y^\beta \, dx\, dy < \infty$$

But then 4) would imply that F_L is bounded on $\{y > \delta\}$ for all $\delta > 0$. Thus $\int F_L(x,y)\,dx \nearrow$ as $y \to 0$, and th

$\beta = 1$.

<u>Case 2</u>. $\beta < -n-1$. Inequality 2) implies that $|u(x,y)| \leq A y^{\alpha}$ $(\alpha > 0)$. If we define:

$$V(x,y) = \begin{cases} u(x,y) & y > 0 \\ -u(x,-y) & y < 0 \\ 0 & y = 0 \end{cases}$$

then V is clearly harmonic in $\mathbb{R}^{n+1}$. Also, $|V| \leq A|y|^{\alpha}$. Thus $V \in \mathcal{S}'$. But $|\xi|^2 \hat{V}(\xi) = 0$ in $\mathcal{S}'$. Therefore $\operatorname{supp} \hat{V} = \{0\}$. Therefore V is a polynomial, which contradicts 3). So $u \equiv 0$. (This argument is due to David Ullrich.)

Our proof of Theorem 2 has a curious consequence. Observe that, while we only require smoothness for certain p and β, the proof yields smooth atoms for all p and β. We thus have the following:

<u>Corollary</u>. Let $0 < p \leq 1$, $\beta > -1$, with either $\beta > 0$ or $\beta = 0$ and $p < 1$. If $f \in D$

satisfies:

$$f = \sum \lambda_k a_k \quad \text{in } \mathcal{S}'$$

where the a_k are (n,p,β)-atoms and $\sum |\lambda_k|^p < \infty$; then there exist γ_k and (n,p,β) atoms b_k, but which also satisfy iv), such that:

$$f = \sum \gamma_k b_k \quad \text{in } \mathcal{S}'$$

and:

$$\sum |\gamma_k|^p \leq C_{p,\beta} \sum |\lambda_k|^p$$

To see what this means, consider an example: Let $f = \sum \lambda_k a_k$, where the a_k are Hardy space H^p atoms (see [3] for the definition), for $p<1$, and $\sum |\lambda_k| < \infty$. There exist smooth H^p atoms b_k and numbers γ_k so that $f = \sum \gamma_k b_k$ and $\sum |\gamma_k| \leq C_p \sum |\lambda_k|$.

We are not sure what (if any) the use of such a result might be.

References

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